Lecture 13
The group SU(2)

The Lie group SU(2) consists of all complex (2×2)-matrices A such that $A\bar{A}^{\top} = E$ and $\det A = 1$. As a manifold, it can be identified with the 3-sphere S^3 as follows: Any matrix $A \in \mathrm{SU}(2)$ is invertible, therefore, the condition $A\bar{A}^{\top} = E$ can be rewritten in the form $\bar{A}^{\top} = A^{-1}$. If

$$A = \begin{pmatrix} a & b \\ u & v \end{pmatrix} \quad \text{then} \quad \bar{A}^{\top} = \begin{pmatrix} \bar{a} & \bar{u} \\ \bar{b} & \bar{v} \end{pmatrix} \quad \text{and} \quad A^{-1} = \begin{pmatrix} v & -b \\ -u & a \end{pmatrix},$$

so the condition $\bar{A}^{\top} = A^{-1}$ is equivalent to $v = \bar{a}$ and $u = -\bar{b}$. Thus any $A \in \mathrm{SU}(2)$ is of the form

$$A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \quad \text{with} \quad a, b \in \mathbb{C}, \quad \det A = |a|^2 + |b|^2 = 1.$$

This provides the identification

$$\mathrm{SU}(2) = \{\, (a, b) \in \mathbb{C}^2 \mid |a|^2 + |b|^2 = 1 \,\} = S^3.$$

On the other hand, SU(2) can be identified with the Lie group Sp(1) of unit quaternions. Recall that the norm $|q|$ of a quaternion $q = x + yi + zj + wk \in \mathbb{H}$ is defined by the formula $|q|^2 = x^2 + y^2 + z^2 + w^2$, or by $|q|^2 = q\bar{q}$ where $\bar{q} = x - yi - zj - wk$. One can easily check that $|pq| = |p||q|$. The group Sp(1) consists of all quaternions q with $|q| = 1$. Since $|x + yi + zj + wk|^2 = x^2 + y^2 + z^2 + w^2$, the group Sp(1) is topologically a 3-sphere. The identification $\mathrm{Sp}(1) = \mathrm{SU}(2)$ at the level of Lie groups is given by the formula

$$a + bj \mapsto \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}. \tag{13.1}$$

The unit quaternions $1, i, j, k$ are identified via (13.1) with the following matrices:

$$1 = E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Let $\mathrm{U}(1) = S^1$ be the group of unit complex numbers. Then the identification (13.1) provides a canonical inclusion $\mathrm{U}(1) \to \mathrm{SU}(2)$ such that

$$e^{i\varphi} \mapsto \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}.$$

Theorem 13.1. *The group* $U(1) \subset SU(2)$ *is a maximal commutative subgroup in* $SU(2)$. *All other maximal commutative subgroups in* $SU(2)$ *are of the form* $C^{-1} \cdot U(1) \cdot C$ *where* $C \in SU(2)$.

This is an easy exercise involving the quaternions. We refer to the elements of $U(1) \subset SU(2)$ as complex numbers.

The trace of a matrix defines a function $\mathrm{tr}\colon SU(2) \to [-2, 2]$, $A \mapsto \mathrm{tr}\, A$. Note that $\pm E$ are the only two matrices in $SU(2)$ with trace ± 2, the rest of the group satisfies $-2 < \mathrm{tr}\, A < 2$.

Theorem 13.2. *Two matrices, A and A', in* $SU(2)$ *are conjugate if and only if* $\mathrm{tr}\, A = \mathrm{tr}\, A'$.

Proof. The $\Rightarrow$ direction is obvious. Consider a matrix $A \in SU(2)$ as a linear operator on $\mathbb{C}^2$. Since $\mathbb{C}$ is algebraically closed, A has an eigenspace with eigenvalue $\lambda \in \mathbb{C}$. Choose a unit vector $\psi = (x, y)$ in this eigenspace, and let

$$C = \begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix} \in SU(2) \quad \text{so that} \quad C\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

then

$$C^{-1}AC\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda \\ 0 \end{pmatrix} \quad \text{or} \quad C^{-1}AC = \begin{pmatrix} \lambda & \alpha \\ 0 & \beta \end{pmatrix}$$

for some $\alpha, \beta \in \mathbb{C}$. Since $C^{-1}AC \in SU(2)$, we have that

$$C^{-1}AC = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix} \quad \text{with} \quad \det C^{-1}AC = |\lambda|^2 = 1.$$

Thus any matrix $A \in SU(2)$ can be conjugated in $SU(2)$ to a matrix of the form

$$\begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} \tag{13.2}$$

whose trace equals $2\cos\varphi$. The trace uniquely defines the angle φ up to sign. Since

$$\begin{pmatrix} e^{-i\varphi} & 0 \\ 0 & e^{i\varphi} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

we are finished. □

In the quaternionic language, this theorem asserts that any unit quaternion is conjugate to a complex number $e^{i\varphi}$, $0 \le \varphi \le \pi$. Thus, the conjugacy classes in $SU(2)$ are in one-to-one correspondence with the sets $\mathrm{tr}^{-1}(c)$ with $-2 \le c \le 2$. The equation

$$\mathrm{tr}\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} = c \tag{13.3}$$

is equivalent to the equation $2\,\mathrm{Re}\,a = c$. The latter defines a hyperplane in $\mathbb{R}^4 = \mathbb{H}$ whose intersection with $\mathrm{SU}(2) = S^3 \subset \mathbb{R}^4$ is $\mathrm{tr}^{-1}(c)$. Thus $\mathrm{tr}^{-1}(c) = S^2$ if $-2 < c < 2$, and $\mathrm{tr}^{-1}(-2) = \{-E\}$, $\mathrm{tr}^{-1}(2) = \{E\}$. Schematically, the conjugacy classes in SU(2) can be pictured as the vertical line segments in Figure 13.1, each segment representing a copy of S^2, which intersects the circle $U(1)$ of unit complex numbers in exactly two points $e^{i\varphi}$, unless $e^{i\varphi} = \pm 1$.

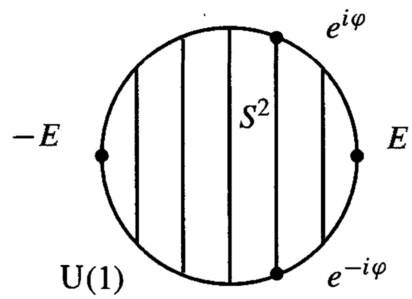

Figure 13.1

At the level of linear spaces, the Lie algebra $\mathfrak{su}(2)$ of the group SU(2) can be identified with the tangent space at 1, so $\mathfrak{su}(2) = T_1\,\mathrm{SU}(2)$. To describe $\mathfrak{su}(2)$ in terms of matrices, we consider the exponential map $\exp\colon T_1\,\mathrm{SU}(2) \to \mathrm{SU}(2)$ given by $\alpha \mapsto e^{\alpha}$. Then $\alpha \in \mathfrak{su}(2)$ if and only if $\exp(\alpha) \in \mathrm{SU}(2)$, which implies the following:

$$\det e^{\alpha} = e^{\mathrm{tr}\,\alpha} = 1 \Rightarrow \mathrm{tr}\,\alpha = 0,$$
$$\overline{(e^{\alpha})}^{\top} = (e^{\alpha})^{-1} \Rightarrow \alpha + \bar{\alpha}^{\top} = 0.$$

Thus, $\mathfrak{su}(2)$ is the 3-dimensional linear space of skew-hermitian matrices with zero trace. All such matrices are of the form

$$\alpha = \begin{pmatrix} ia & b \\ -\bar{b} & -ia \end{pmatrix}, \quad a \in \mathbb{R}, \quad b \in \mathbb{C}.$$

If $b = u + iv$ with $u, v \in \mathbb{R}$ then

$$\begin{pmatrix} ia & u+iv \\ -u+iv & -ia \end{pmatrix} = a\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + u\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + v\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

corresponds to the quaternion $ai + uj + vk$. Therefore, $\mathfrak{su}(2)$ can be thought of as consisting of purely imaginary quaternions. The Lie algebra structure on $\mathfrak{su}(2)$ is given by the Lie bracket

$$[\alpha, \beta] = \alpha\beta - \beta\alpha. \tag{13.4}$$

Theorem 13.3. *The exponential map provides a diffeomorphism*

$$\exp\colon B_{\pi}(0) \to \mathrm{SU}(2) \setminus \{-E\}$$

where $B_{\pi}(0) \subset \mathfrak{su}(2)$ is the open ball of radius π centered at the origin.

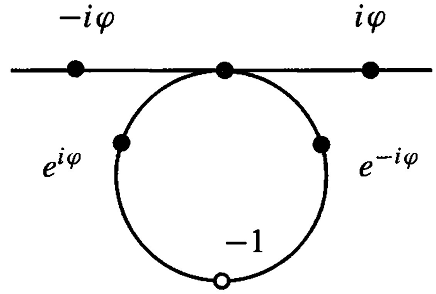

Figure 13.2

Proof. As $e^{g\alpha g^{-1}} = ge^{\alpha}g^{-1}$ we only need to show that the map $\exp\colon (-i\pi, i\pi) \to S^1 \setminus \{-1\}$, $i\varphi \mapsto e^{i\varphi}$, is a diffeomorphism. The latter is obvious, see Figure 13.2. □

Remark. The following is a useful formula for evaluating the exponential map. Let q be a purely imaginary quaternion of unit length, so that $\operatorname{Re} q = 0$ and $|q|^2 = 1$. Then, for any real number θ,

$$e^{q\theta} = \cos\theta + q \cdot \sin\theta.$$

This can be seen as follows. Since $\operatorname{Re} q = 0$, there exists a unit quaternion u such that $q = uiu^{-1}$. Then

$$e^{q\theta} = e^{uiu^{-1}\theta} = e^{u(i\theta)u^{-1}} = ue^{i\theta}u^{-1} = u(\cos\theta + i\sin\theta)u^{-1} = \cos\theta + q \cdot \sin\theta.$$

The tangent space $T_g\,\mathrm{SU}(2)$ at $g \in \mathrm{SU}(2)$ can be identified with the image of $\mathfrak{su}(2)$ under left translation by g:

$$(L_g)_*(\mathrm{SU}(2)) = T_g\,\mathrm{SU}(2),$$

where $L_g(u) = g \cdot u$.

Example. The space $T_i\,\mathrm{SU}(2)$ consists of the matrices of the form

$$i\begin{pmatrix} ia & b \\ -\bar{b} & -ia \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} ia & b \\ -\bar{b} & -ia \end{pmatrix} = \begin{pmatrix} -a & ib \\ i\bar{b} & -a \end{pmatrix}.$$

The group SU(2) acts on itself by conjugation,

$$A \mapsto \mathrm{Ad}_A\colon \mathrm{SU}(2) \to \mathrm{SU}(2), \quad \mathrm{Ad}_A(B) = ABA^{-1}.$$

For any A, the derivative $d_1\,\mathrm{Ad}_A$ of Ad_A at 1 gives an action of SU(2) on its Lie algebra, called again Ad_A,

$$A \mapsto \mathrm{Ad}_A\colon \mathfrak{su}(2) \to \mathfrak{su}(2), \quad \mathrm{Ad}_A(\alpha) = A\alpha A^{-1}.$$

Note that Ad_A is a Lie algebra homomorphism with respect to the bracket (13.4). Thus we have a homomorphism

$$\mathrm{Ad}\colon\ \mathrm{SU}(2) \to \mathrm{Aut}(\mathfrak{su}(2)), \quad A \mapsto \mathrm{Ad}_A\,. \tag{13.5}$$

The derivative of this map at $1 \in \mathrm{SU}(2)$ can be computed as follows: choose $\alpha \in \mathfrak{su}(2)$, then, up to order ε^2,

$$\mathrm{Ad}_{1+\varepsilon\alpha}(\beta) = (1+\varepsilon\alpha)\beta(1-\varepsilon\alpha) = \beta + \varepsilon(\alpha\beta - \beta\alpha).$$

Denote by $\mathrm{ad}_\alpha\colon \mathfrak{su}(2) \to \mathfrak{su}(2)$ the operator $\mathrm{ad}_\alpha(\beta) = [\alpha, \beta]$, then $\mathrm{Ad}_{1+\varepsilon\alpha}(\beta) = (1+\varepsilon\,\mathrm{ad}_\alpha)(\beta)$ up to order ε^2. Thus, $(d_1\,\mathrm{Ad})(\alpha) = \mathrm{ad}_\alpha$.

Theorem 13.4. *The map* (13.5) *is well-defined as a Lie group homomorphism* $\mathrm{SU}(2) \to \mathrm{SO}(3)$. *It is the universal (double) cover of* $\mathrm{SO}(3)$, *hence* $\pi_1\,\mathrm{SO}(3) = \mathbb{Z}/2$.

Proof. The map (13.5) is a homomorphism; $\mathrm{Ad}(AB) = \mathrm{Ad}(A)\,\mathrm{Ad}(B)$ because $\mathrm{Ad}_{AB}(x) = ABx(AB)^{-1} = A(BxB^{-1})A^{-1} = \mathrm{Ad}_A\,\mathrm{Ad}_B(x)$. Since $\mathfrak{su}(2) = \mathbb{R}^3$ as a linear space, one can think of $\mathrm{Aut}(\mathfrak{su}(2))$ as a subgroup of $\mathrm{GL}_3(\mathbb{R})$. Then the map (13.5) is well-defined as a homomorphism $\mathrm{Ad}\colon\ \mathrm{SU}(2) \to \mathrm{GL}_3(\mathbb{R})$.

The Euclidean dot-product in $\mathbb{R}^3 = \mathfrak{su}(2)$ can be described by the formula

$$u \cdot v = -\frac{1}{2}\,\mathrm{tr}(uv) = -\,\mathrm{Re}(uv)$$

depending on the realization of $\mathfrak{su}(2)$ by either matrices or quaternions. One can easily check that Ad_A preserves the dot-product:

$$\begin{aligned}(\mathrm{Ad}_A\,u)\cdot(\mathrm{Ad}_A\,v) &= -\frac{1}{2}\,\mathrm{tr}(AuA^{-1}\,AvA^{-1})\\ &= -\frac{1}{2}\,\mathrm{tr}(uv) = u\cdot v.\end{aligned}$$

Therefore, $\mathrm{Ad}(\mathrm{SU}(2)) \subset \mathrm{O}(3)$, the orthogonal group of $\mathbb{R}^3$. Since $\mathrm{SU}(2)$ is connected, the image of $\mathrm{SU}(2)$ should belong to the connected component of the identity in $\mathrm{O}(3)$, that is, to $\mathrm{SO}(3)$.

The map $\mathrm{SU}(2) \to \mathrm{SO}(3)$ is surjective. Each matrix from $\mathrm{SO}(3)$, thought of as acting on $\mathbb{R}^3$, is a product of rotations about the coordinate axes. Thus to show surjectivity we only need to show that the matrix

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{pmatrix}$$

of rotation about the x-axis through an angle ψ belongs to the image of Ad. The rotations with respect to the other two coordinate axes can be handled similarly. Let $\varphi = \psi/2$ and

$$A = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}$$

then

$$\begin{aligned} \mathrm{Ad}_A(i) &= e^{i\varphi} i e^{-i\varphi} = i, \\ \mathrm{Ad}_A(j) &= e^{i\varphi} j e^{-i\varphi} = e^{2i\varphi} j = \cos\psi \cdot j + \sin\psi \cdot k, \\ \mathrm{Ad}_A(k) &= e^{i\varphi} k e^{-i\varphi} = e^{2i\varphi} k = \cos\psi \cdot k - \sin\psi \cdot j. \end{aligned}$$

Therefore, $\mathrm{Ad}_A = R_x$.

Suppose that $\mathrm{Ad}_A = \mathrm{Ad}_B$ then $ACA^{-1} = BCB^{-1}$ for all $C \in \mathrm{SU}(2)$, in other words, $B^{-1}A$ belongs to the center of SU(2). Since the center of SU(2) consists of $\pm E$, we get $B = \pm A$. Hence Ad is a double cover. □

Algebraically, the homomorphism $\mathrm{SU}(2) \to \mathrm{SO}(3)$ can be described as the quotient map of SU(2) by its center $\mathbb{Z}/2 = \{\pm E\}$. Topologically, it is the standard double cover $S^3 \to \mathbb{R}P^3$ after the identification $\mathrm{SO}(3) = \mathbb{R}P^3$.